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A generalization of Pascal's mystic hexagram

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Resum *(CAT)*
El teorema clàssic de Pascal afirma que si un hexàgon a $\mathbb{P}^2(\mathbb{C})$ està inscrit en una cònica llavors els costats oposats de l'hexàgon es troben en tres punts que s'ubiquen sobre una recta, anomenada recta de Pascal. Zhongxuan Luo va donar el 2007 una generalització del teorema de Pascal per a corbes de grau arbitrari. En el present article es donen dues demostracions d'aquesta generalització. La primera és autocontiguda i fa ús del teorema de Carnot, mentre que la segona es basa en el teorema fonamental de Max Noether.

Abstract *(ENG)*
Pascal's classical theorem asserts that if a hexagon in $\mathbb{P}^2(\mathbb{C})$ is inscribed in a conic, then the opposite sides of the hexagon lie on a straight line, called Pascal line. Zhongxuan Luo gave in 2007 a generalization of Pascal's theorem for curves of arbitrary degree. In the present article, two proofs of this generalization are given. The first one is self-contained and makes use of Carnot's theorem, while the second proof is based on Max Noether's Fundamental theorem.

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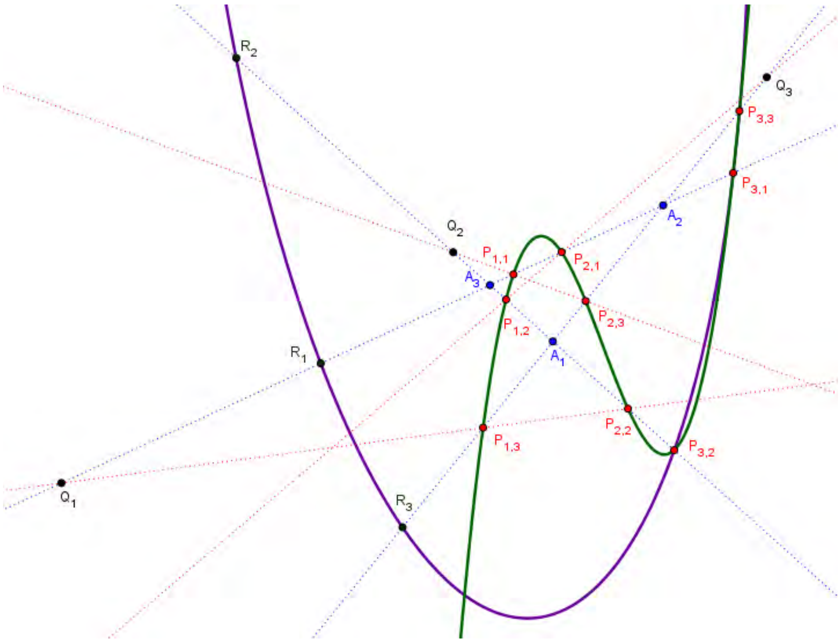

Figure 1: Pascal Type theorem for n = 3.

1. Introduction

One of the classical results in projective geometry is Pascal's theorem, also known as Pascal's Mystic Hexagram. This theorem, obtained by Blaise Pascal in 1640, asserts that if a hexagon in $\mathbb{P}^2(\mathbb{C})$ is inscribed in a conic, then the opposite sides of the hexagon lie on a straight line; c.f. [6, § 5.6, Cor. 1].

There are many known generalizations of Pascal's theorem. For example, Chasles' theorem (c.f. [3]) or the Cayley-Bacharach theorem (c.f. [5]) are generalizations of Pascal's theorem. In [8], Zhongxuan Luo presents another generalization of Pascal's theorem (see Fig. 1): *Let l_1, l_2, l_3 be three non-concurrent lines and take a collection of $n \geq 2$ points $S_i \subset l_i$ on each line, such that $S_i \cap l_j = \emptyset$ for $j \neq i$. Choose two points $P_{1,i}, P_{2,i} \in S_i$ on each collection and let R_1, R_2, R_3 be the triple of points given by the Pascal mapping (see Definition 2.11) applied to the six chosen points. Then the $3n$ points $S_1 \cup S_2 \cup S_3$ lie on an algebraic curve of degree n that contains none of the lines l_1, l_2, l_3 if and only if there exists an algebraic curve of degree $n-1$ intersecting each line l_i in $\{R_i\} \cup S_i \setminus \{P_{1,i}, P_{2,i}\}$.*

The aim of this article is to present two different proofs of Zhongxuan Luo's extension of Pascal's theorem. The first proof is elementary and makes use of a version of Carnot's theorem; see Section 3. The approach is similar to [8], but we do not use spline theory. The second proof is based on Max Noether's Fundamental theorem; see Section 4.

Throughout this paper we work in the complex projective plane $\mathbb{P}^2(\mathbb{C})$ and we set a projective reference $R = \{A_1, A_2, A_3; O\}$, so that $A_1 = (1:0:0)$, $A_2 = (0:1:0)$, $A_3 = (0:0:1)$ and $O = (1:1:1)$. Observe that then $A_{2,3} = OA_1 \cap A_2A_3 = (0:1:1)$, $A_{3,1} = OA_2 \cap A_3A_1 = (1:0:1)$, and $A_{1,2} = OA_3 \cap A_1A_2 = (1:1:0)$, where for points $A, B \in \mathbb{P}^2(\mathbb{C})$, we mean AB to be the projective line that joins A and B. Moreover, we set $l_1 = A_2A_3$, $l_2 = A_3A_1$ and $l_3 = A_1A_2$ to be the sides of the projective triangle

$A_1A_2A_3$ with vertices at points A_1, A_2, A_3.

2. An extension of Carnot's theorem

In this section we review a generalization of Carnot's and Menelaus's theorems, which allows to determine whether a certain configuration of points lies on an algebraic curve of a given degree. We also introduce some constructions from [8] that appear in the generalization of Pascal's theorem.

Definition 2.1. Given points $P_1, \dots, P_r \in A_iA_j \backslash \{A_i, A_j\}$, where $i, j \in \{1, 2, 3\}$, $i \neq j$. We define the characteristic ratio of $P_1, \dots, P_r$ with respect to the reference R to be $[A_i, A_j; P_1, \dots, P_r]_R = \prod_{k=1}^{r}(A_i, A_j, A_{i,j}, P_k)$, where $(A_i, A_j, A_{i,j}, P_k)$ denotes the cross ratio; c.f. [4, § 5.2].

The notion of characteristic ratio defined in [8] and the one defined in Definition 2.1 are inverse to each other.

Example 2.2. Let $P_k = (0 : \lambda_k : 1) \in A_2A_3$ with $\lambda_k \in \mathbb{C} \setminus \{0\}$, $k = 1, \dots, r$. Then, $[A_2, A_3; P_1, \dots, P_r]_R = \prod_{k=1}^{r}(A_2, A_3, A_{2,3}, P_k) = \prod_{k=1}^{r} \lambda_k$.

With this notation at hand, Menelaus's theorem (c.f. [7]) and Carnot's theorem (c.f. [2]) can be stated as follows.

Theorem 2.3 (Menelaus's Theorem)**.** *Let $P_i \in l_i$, $i = 1, 2, 3$, be points different from A_1, A_2 and A_3. Then, P_1, P_2 and P_3 are collinear if and only if $[A_2, A_3; P_1]_R[A_3, A_1; P_2]_R[A_1, A_2; P_3]_R = -1$.*

Theorem 2.4 (Carnot's Theorem)**.** *Let $P_1, P_2 \in l_1$, $P_3, P_4 \in l_2$, and $P_5, P_6 \in l_3$ be six distinct points different from A_1, A_2 and A_3. Then, $P_1, P_2, \dots, P_6$ lie on a conic disjoint with $\{A_1, A_2, A_3\}$ if and only if $[A_2, A_3; P_1, P_2]_R[A_3, A_1; P_3, P_4]_R[A_1, A_2; P_5, P_6]_R = 1$.*

The next theorem is a natural generalization of Menelaus's and Carnot's theorems to curves of arbitrary degree. It is called Carnot's theorem in [1] and is equivalent to [8, Thm. 4.4]. For completeness we provide a proof here.

Theorem 2.5. *Let $S_i = \{P_{1,i}, \dots, P_{n,i}\}$ be a collection of n different points of $l_i \setminus \{A_1, A_2, A_3\}$, $i = 1, 2, 3$. Then, $S_1 \cup S_2 \cup S_3$ lie on an algebraic curve of degree n disjoint with $\{A_1, A_2, A_3\}$ if and only if*

$$[A_2, A_3; P_{1,1}, \dots, P_{n,1}]_R[A_3, A_1; P_{1,2}, \dots, P_{n,2}]_R[A_1, A_2; P_{1,3}, \dots, P_{n,3}]_R = (-1)^n.$$

Proof. Recall that the cases $n = 1$ and $n = 2$ are Menelaus's theorem and Carnot's theorem respectively. Then we can assume that $n \geq 3$. We denote

$$\mathbb{C}[X, Y, Z]_n = \{F \in \mathbb{C}[X, Y, Z];\ F \text{ homogeneous polynomial of degree } n\}.$$

With this in hand, we define the map $\varphi \colon \mathbb{C}[X, Y, Z]_n \big/ (XYZ) \to \mathbb{C}[Y, Z]_n \times \mathbb{C}[X, Z]_n \times \mathbb{C}[X, Y]_n$ such that $\varphi([F(X, Y, Z)]) = (F(0, Y, Z), F(X, 0, Z), F(X, Y, 0))$.

Clearly, φ is well defined and linear. Let us see that it is also injective: if $[F] \in ker(\varphi)$, then $\varphi([F(X, Y, Z)]) = (F(0, Y, Z), F(X, 0, Z), F(X, Y, 0)) = (0, 0, 0)$. Thus, X, Y and Z divide F. Therefore, $[F] = [0]$.

Hence, φ is an isomorphism over its image. We claim that the image of φ is exactly the set $M_n \subset \mathbb{C}[Y,Z]_n \times \mathbb{C}[X,Z]_n \times \mathbb{C}[X,Y]_n$ defined as

$$M_n := \left\{ \left(\sum_{i+j=n} B_{i,j} Y^i Z^j, \sum_{i+j=n} C_{i,j} X^i Z^j, \sum_{i+j=n} D_{i,j} X^i Y^j \right) ; B_{n,0} = D_{0,n}, B_{0,n} = C_{0,n}, C_{n,0} = D_{n,0} \right\}.$$

Clearly, $Im(\varphi) \subseteq M_n$. Moreover, by computing dimensions, we find

$$\begin{aligned} \dim_{\mathbb{C}} \left(\mathbb{C}[X,Y,Z]_n \big/ (XYZ) \right) = 3n &= \dim_{\mathbb{C}}(\mathbb{C}[Y,Z]_n) + \dim_{\mathbb{C}}(\mathbb{C}[X,Z]_n) \\ &\quad + \dim_{\mathbb{C}}(\mathbb{C}[X,Y]_n) - 3 = \dim_{\mathbb{C}}(M_n). \end{aligned}$$

It follows that $Im(\varphi) = M_n$.

Let $P_{i,1} = (0 : a_i : 1)$, $P_{i,2} = (1 : 0 : b_i)$, and $P_{i,3} = (c_i : 1 : 0)$ with $a_i, b_i, c_i \in \mathbb{C} \setminus \{0\}$, $i = 1, 2, \ldots, n$. Note that, by degree reasons, any curve of degree n containing $S_1 \cup S_2 \cup S_3$ and not containing l_1, l_2 nor l_3 must be disjoint with $\{A_1, A_2, A_3\}$. Such a curve exists if and only if

$$\left(\lambda_1 \prod_{i=1}^{n} (a_i Z - Y),\ \lambda_2 \prod_{i=1}^{n} (b_i X - Z),\ \lambda_3 \prod_{i=1}^{n} (c_i Y - X) \right) \in M_n, \tag{1}$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} \setminus \{0\}$. According to the definition of M_n, a necessary and sufficient condition for (1) to be true is that the following system

$$\begin{cases} \lambda_1 \prod_{i=1}^{n} a_i &= (-1)^n \lambda_2, \\ (-1)^n \lambda_1 &= \lambda_3 \prod_{i=1}^{n} c_i, \\ \lambda_2 \prod_{i=1}^{n} b_i &= (-1)^n \lambda_3, \end{cases} \tag{2}$$

has a non-trivial solution for $\lambda_1, \lambda_2, \lambda_3$. However, the system (2) has a non-trivial solution if and only if $(-1)^n = \prod_{i=1}^{n} a_i b_i c_i = [A_2, A_3; P_{1,1}, \ldots, P_{n,1}]_R [A_3, A_1; P_{1,2}, \ldots, P_{n,2}]_R [A_1, A_2; P_{1,3}, \ldots, P_{n,3}]_R$. This completes the proof. □

Next, we introduce some notions from [8] that will be needed in the generalization of Pascal's theorem.

Definition 2.6. The characteristic map $\sigma_{i,j} \colon A_iA_j \to A_iA_j$ relative to $A_i, A_j, A_{i,j}$ is the projective involution that satisfies $\sigma_{i,j}(A_i) = A_j$, $\sigma_{i,j}(A_j) = A_i$, and $\sigma_{i,j}(A_{i,j}) = A_{i,j}$, where $i, j = 1, 2, 3$, $i \neq j$.

Observation 2.7. If $P = \sigma_{i,j}(Q)$ is the image of Q under the characteristic map relative to $A_i, A_j, A_{i,j}$, then $[A_i, A_j; P, Q]_R = (A_i, A_j, A_{i,j}, P)(A_i, A_j, A_{i,j}, Q) = 1$.

An interesting fact is that if we take a point in each side of a triangle, and we consider their respective images of the characteristic map relative to each side, then these six points lie on a conic. This property will take an important role in Section 4.

Proposition 2.8. *Let $P_1 \in l_1$, $P_2 \in l_2$ and $P_3 \in l_3$ be three points different from A_1, A_2, A_3. Then, P_1, P_2, P_3, $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$ and $\sigma_{1,2}(P_3)$ lie on a conic.*

Proof. By Observation 2.7, $[A_2, A_3; \sigma_{2,3}(P_1), P_1]_R = [A_3, A_1; \sigma_{3,1}(P_2), P_2]_R = [A_1, A_2; \sigma_{1,2}(P_3), P_3]_R = 1$. Then, the result follows by Carnot's theorem. □

The following results contain basic properties of the characteristic map; c.f. [8, Cor. 3.7, 3.9].

Proposition 2.9. *Any three distinct points $P_1 \in l_1$, $P_2 \in l_2$, and $P_3 \in l_3$ different from A_1, A_2, A_3 are collinear if and only if $\sigma_{2,3}(P_1)$, $\sigma_{3,1}(P_2)$, and $\sigma_{1,2}(P_3)$ are collinear.*

Proof. Let $P_1 = (0 : a : 1)$, $P_2 = (1 : 0 : b)$, and $P_3 = (c : 1 : 0)$, with $a, b, c \in \mathbb{C} \setminus \{0\}$. Then, $\sigma_{2,3}(P_1) = (0 : 1 : a)$, $\sigma_{3,1}(P_2) = (b : 0 : 1)$, and $\sigma_{1,2}(P_3) = (1 : c : 0)$. Menelaus's theorem asserts that a necessary and sufficient condition of P_1, P_2, P_3 to be collinear is that

$$-1 = [A_2, A_3; P_1]_R[A_3, A_1; P_2]_R[A_1, A_2; P_3]_R = abc. \tag{3}$$

Similarly, their images under their corresponding characteristic map are collinear if and only if

$$-1 = [A_2, A_3; \sigma_{2,3}(P_1)]_R[A_3, A_1; \sigma_{3,1}(P_2)]_R[A_1, A_2; \sigma_{1,2}(P_3)]_R = \frac{1}{abc}. \tag{4}$$

Since both equalities (3) and (4) are equivalent, this completes the proof. □

Proposition 2.10. *Let $P_1, P_2 \in l_1$, $P_3, P_4 \in l_2$, and $P_5, P_6 \in l_3$ be any six distinct points different from A_1, A_2, A_3. Then, $P_1, P_2, \dots, P_6$ lie on a conic if and only if their images by the corresponding characteristic map lie on a conic as well.*

Proof. Let $P_1 = (0 : a_1 : 1)$, $P_2 = (0 : a_2 : 1)$, $P_3 = (1 : 0 : b_1)$, $P_4 = (1 : 0 : b_2)$, $P_5 = (c_1 : 1 : 0)$, and $P_6 = (c_2 : 1 : 0)$, with $a_i, b_i, c_i \in \mathbb{C} \setminus \{0\}$, $i = 1, 2$. Then, $\sigma_{2,3}(P_1) = (0 : 1 : a_1)$, $\sigma_{2,3}(P_2) = (0 : 1 : a_2)$, $\sigma_{3,1}(P_3) = (b_1 : 0 : 1)$, $\sigma_{3,1}(P_4) = (b_2 : 0 : 1)$, $\sigma_{1,2}(P_5) = (1 : c_1 : 0)$, and $\sigma_{1,2}(P_6) = (1 : c_2 : 0)$. By Carnot's theorem, the six points $P_1, P_2, \dots, P_6$ lie on a conic if and only if

$$1 = [A_2, A_3; P_1, P_2]_R[A_3, A_1; P_3, P_4]_R[A_1, A_2; P_5, P_6]_R = a_1a_2b_1b_2c_1c_2. \tag{5}$$

Similarly, their images under their corresponding characteristic map lie on a conic if and only if

$$\begin{aligned} 1 = [A_2, A_3;&\sigma_{2,3}(P_1), \sigma_{2,3}(P_2)]_R[A_3, A_1; \sigma_{3,1}(P_3), \sigma_{3,1}(P_4)]_R \\ &\times [A_1, A_2; \sigma_{1,2}(P_5), \sigma_{1,2}(P_6)]_R = \frac{1}{a_1a_2b_1b_2c_1c_2}. \end{aligned} \tag{6}$$

Since both equalities (5) and (6) are equivalent, this completes the proof. □

The following construction from [8] plays a crucial role in Zhongxuan Luo's generalization of Pascal's theorem.

Definition 2.11. The Pascal mapping is the map $\Psi := (\sigma_{2,3} \times \sigma_{3,1} \times \sigma_{1,2}) \circ \Phi$, where $\Phi\colon (l_1 \setminus \{A_2, A_3\})^2 \times (l_2 \setminus \{A_3, A_1\})^2 \times (l_3 \setminus \{A_1, A_2\})^2 \to l_1 \times l_2 \times l_3$ satisfies

$$\Phi((P_1, P_2), (P_3, P_4), (P_5, P_6)) = \{P_1P_2 \cap P_4P_5, P_3P_4 \cap P_6P_1, P_5P_6 \cap P_2P_3\}.$$

If we denote $Q_1 = P_1P_2 \cap P_4P_5$, $Q_2 = P_3P_4 \cap P_6P_1$, and $Q_3 = P_5P_6 \cap P_2P_3$ then, $\Psi((P_1, P_2), (P_3, P_4), (P_5, P_6)) = \{\sigma_{2,3}(Q_1), \sigma_{3,1}(Q_2), \sigma_{1,2}(Q_3)\}$.

3. A Pascal type theorem

In this section we present the generalization of Pascal's theorem given in [8]. We give an elementary proof based on the results of the previous section. In Section 4 we will give a second proof using Max Noether's Fundamental theorem. First let us recall the complete version of Pascal's original theorem. For completeness, we provide a proof here based on Menelaus's and Carnot's theorems.

Theorem 3.1. *Let $P_1, P_2 \in l_1$, $P_3, P_4 \in l_2$, and $P_5, P_6 \in l_3$ be six distinct points, all of them different from A_1, A_2, A_3, and let $Q_1 = P_1P_2 \cap P_4P_5$, $Q_2 = P_3P_4 \cap P_6P_1$, and $Q_3 = P_5P_6 \cap P_2P_3$. Then, $P_1, P_2, \dots, P_6$ lie on a conic if and only if Q_1, Q_2, Q_3 are collinear.*

Proof. Let $P_1 = (0 : a_1 : 1)$, $P_2 = (0 : a_2 : 1)$, $P_3 = (1 : 0 : b_1)$, $P_4 = (1 : 0 : b_2)$, $P_5 = (c_1 : 1 : 0)$, and $P_6 = (c_2 : 1 : 0)$, with $a_i, b_i, c_i \neq 0$, $i = 1, 2$; it follows that $Q_1 = (0 : -1 : b_2c_1)$, $Q_2 = (a_1c_2 : 0 : -1)$, and $Q_3 = (-1 : b_1a_2 : 0)$. By Carnot's theorem, $P_1, P_2, \dots, P_6$ lie on a conic disjoint from $\{A_1, A_2, A_3\}$ if and only if

$$1 = [A_2, A_3; P_1, P_2]_R[A_3, A_1; P_3, P_4]_R[A_1, A_2; P_5, P_6]_R = a_1a_2b_1b_2c_1c_2. \tag{7}$$

Similarly, by Menelaus's theorem, we have that a necessary and sufficient condition for Q_1, Q_2, Q_3 to be collinear is that

$$-1 = [A_2, A_3; Q_1]_R[A_3, A_1; Q_2]_R[A_1, A_2; Q_3]_R = \frac{-1}{b_2c_1}\frac{-1}{a_1c_2}\frac{-1}{b_1a_2}. \tag{8}$$

Since both equalities (7) and (8) are equivalent, this proves the theorem. $\square$

Notice that by Proposition 2.9, if Q_1, Q_2, Q_3 lie on the same line, then the points in the Pascal mapping $\Psi((P_1, P_2), (P_3, P_4), (P_5, P_6))$ are also collinear points. It is precisely this version of Pascal's theorem that was generalized by Zhongxuan Luo to higher degrees. The precise statement is the following.

Theorem 3.2 (Pascal Type Theorem). *Let $S_j = \{P_{i,j}\}_{i=1}^n$ be a collection of $n \geq 2$ distinct points on the set $l_j \setminus \{A_1, A_2, A_3\}$, $j = 1, 2, 3$. Let us choose two points on each collection S_j, and let R_1, R_2, R_3 be the triple given by the Pascal mapping applied to the six chosen points. Then, the $3n$ points $S_1 \cup S_2 \cup S_3$ lie on an algebraic curve of degree n disjoint with $\{A_1, A_2, A_3\}$ if and only if there exists an algebraic curve of degree $n-1$ disjoint with $\{A_1, A_2, A_3\}$ which contains R_1, R_2, R_3 and the $3(n-2)$ points from $S_1 \cup S_2 \cup S_3$ that have not been chosen.*

Proof. Let us take $a_i, b_i, c_i \in \mathbb{C} \setminus \{0\}$ and $P_{i,1} = (0 : a_i : 1)$, $P_{i,2} = (1 : 0 : b_i)$, and $P_{i,3} = (c_i : 1 : 0)$ for every $i = 1, \dots, n$. Without loss of generality, let us choose the points $P_{1,1}$, $P_{2,1}$, $P_{1,2}$, $P_{2,2}$, $P_{1,3}$ and $P_{2,3}$, to apply the Pascal mapping; see Fig. 1 above. Then,

$$\Psi\left(\left(P_{1,1}, P_{2,1}\right), \left(P_{1,2}, P_{2,2}\right), \left(P_{1,3}, P_{2,3}\right)\right) = \{R_1, R_2, R_3\}, \tag{9}$$

where $R_1 = \sigma_{2,3}(Q_1) = (0 : b_2c_1 : -1)$, $R_2 = \sigma_{3,1}(Q_2) = (-1 : 0 : a_1c_2)$, and $R_3 = \sigma_{1,2}(Q_3) = (b_1a_2 : -1 : 0)$, with $Q_1 = P_{1,1}P_{2,1} \cap P_{2,2}P_{1,3}$, $Q_2 = P_{1,2}P_{2,2} \cap P_{2,3}P_{1,1}$, and $Q_3 = P_{1,3}P_{2,3} \cap P_{2,1}P_{1,2}$.

By Theorem 2.5, the $3n$ points $S_1 \cup S_2 \cup S_3$ lie on an algebraic curve of degree n disjoint with $\{A_1, A_2, A_3\}$ if and only if

$$\begin{aligned}(-1)^n &= [A_2, A_3; P_{1,1}, \dots, P_{n,1}]_R[A_3, A_1; P_{1,2}, \dots, P_{n,2}]_R[A_1, A_2; P_{1,3}, \dots, P_{n,3}]_R \\ &= \prod_{i=1}^n (A_2, A_3, A_{2,3}, P_{i,1})(A_3, A_1, A_{3,1}, P_{i,2})(A_1, A_2, A_{1,2}, P_{i,3}) = \prod_{i=1}^n a_ib_ic_i.\end{aligned} \tag{10}$$

Similarly, there exists an algebraic curve of degree $n-1$ disjoint with $\{A_1, A_2, A_3\}$ which contains R_1, R_2, R_3 and the $3(n-2)$ points from $S_1 \cup S_2 \cup S_3$ that have not been chosen if and only if

$$\begin{aligned}(-1)^{n-1} &= [A_2, A_3; P_{3,1}, \dots, P_{n,1}, R_1]_R [A_3, A_1; P_{3,2}, \dots, P_{n,2}, R_2]_R [A_1, A_2; P_{3,3}, \dots, P_{n,3}, R_3]_R \\ &= \left[(A_2, A_3, A_{2,3}, R_1) \prod_{i=3}^{n} (A_2, A_3, A_{2,3}, P_{i,1})\right] \left[(A_3, A_1, A_{3,1}, R_2) \prod_{i=3}^{n} (A_3, A_1, A_{3,1}, P_{i,2})\right] \\ &\times \left[(A_1, A_2, A_{1,2}, R_3) \prod_{i=3}^{n} (A_1, A_2, A_{1,2}, P_{i,3})\right] = -\prod_{i=1}^{n} a_i b_i c_i.\end{aligned} \tag{11}$$

Since both equalities (10) and (11) are equivalent, this completes the proof. □

4. A Pascal type theorem and Max Noether's fundamental theorem

We give a new proof of Theorem 3.2 based on Max Noether's Fundamental theorem; in particular, we will make us of a corollary of it. To do so, we need a few basic notions about algebraic curves in $\mathbb{P}^2(\mathbb{C})$; for more details, see [6].

Max Noether's Fundamental theorem is concerned with the following question (c.f. [6, § 5.5]): suppose C, C' are two projective plane curves with no common factors, and C'' is another curve satisfying $C \cap C' \subset C \cap C''$, when counted with multiplicity. So, when is there a curve that intersects C in the points of $C \cap C''$ that are not in $C \cap C'$?

For our purpose, we do not use directly Max Noether's Fundamental theorem, but we use a corollary of it. First, if C, C' are projective plane curves with no common components, the intersection cycle $C \cdot C'$ is defined as the formal sum

$$C \cdot C' = \sum_{P \in C \cap C'} m_P(C, C')P,$$

where $m_P(C, C')$ is the multiplicity of the point P in $C \cap C'$; c.f. [6, § 5.5]. In particular, $m_P(C, C') = 0$ if and only if $P \notin C \cap C'$.

If C, C' are projective plane curves of degree n and m respectively, CC' is the projective plane curve of degree $n + m$ consisting on the union of C and C'.

With this notation at hand, we are in conditions to state the corollary of Max Noether's Fundamental theorem; c.f. [6, § 5.5, Cor. 2].

Theorem 4.1. *Let C, C', C'' be projective plane curves such that C' and C'' have no common component with C. If all the points of $C \cap C'$ are simple points of C and $C \cdot C'' \geq C \cdot C'$ (i.e., $m_P(C, C'') \geq m_P(C, C')$ for every $P \in C$), then there is a curve Γ of degree $deg(\Gamma) = deg(C'') - deg(C')$ such that $C \cdot \Gamma = C \cdot C'' - C \cdot C'$.*

Now we are in conditions to give the new proof of Theorem 3.2.

Proof. (Pascal Type Theorem) Let us take the same notation as in (9). Here is when Proposition 2.8 becomes crucial, since it asserts that the points $Q_1, Q_2, Q_3, R_1, R_2, R_3$ lie on a conic; let Γ_2 be that conic.

Let $C = (P_{2,1}P_{1,2})(P_{2,2}P_{1,3})(P_{2,3}P_{1,1})$ be the cubic generated by the three opposites sides, disjoint with $\{A_1, A_2, A_3\}$, of the hexagon with vertices $P_{1,1}, P_{2,1}, P_{1,2}, P_{2,2}, P_{1,3}, P_{2,3}$.

First assume that the $3n$ points $S_1 \cup S_2 \cup S_3$ lie on an algebraic curve Γ_n of degree n disjoint with $\{A_1, A_2, A_3\}$ and consider the algebraic curve $\Gamma_{n+2} = \Gamma_2\Gamma_n$ of degree $n+2$. Then, we have that

$$\Gamma_{n+2} \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_{i,1} + P_{i,2} + P_{i,3}) + Q_1 + Q_2 + Q_3 + R_1 + R_2 + R_3,$$

and $C \cdot l_1 l_2 l_3 = P_{1,1} + P_{2,1} + P_{1,2} + P_{2,2} + P_{1,3} + P_{2,3} + Q_1 + Q_2 + Q_3$. Therefore, $\Gamma_{n+2} \cdot l_1 l_2 l_3 - C \cdot l_1 l_2 l_3 = \sum_{i=3}^{n} (P_{i,1} + P_{i,2} + P_{i,3}) + R_1 + R_2 + R_3$.

By Theorem 4.1, there exists a curve Γ of degree $\deg(\Gamma) = \deg(\Gamma_{n+2}) - \deg(C) = n-1$ such that $\Gamma \cdot l_1 l_2 l_3 = \sum_{i=3}^{n} (P_{i,1} + P_{i,2} + P_{i,3}) + R_1 + R_2 + R_3$. So, Γ is an algebraic curve of degree $n-1$ that passes through the $3(n-1)$ points $P_{3,1}, \dots, P_{n,1}, P_{3,2}, \dots, P_{n,2}, P_{3,3}, \dots, P_{n,3}, R_1, R_2, R_3$.

Reciprocally, suppose that there exists an algebraic curve Γ'_{n-1} of degree $n-1$ disjoint with $\{A_1, A_2, A_3\}$ that contains R_1, R_2, R_3 and the $3(n-2)$ points from $S_1 \cup S_2 \cup S_3$ that have not been chosen. Consider the algebraic curve $\Gamma'_{n+2} = C\Gamma'_{n-1}$ of degree $n+2$. Then, we have that

$$\Gamma'_{n+2} \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_{i,1} + P_{i,2} + P_{i,3}) + Q_1 + Q_2 + Q_3 + R_1 + R_2 + R_3,$$

and $\Gamma_2 \cdot l_1 l_2 l_3 = Q_1 + Q_2 + Q_3 + R_1 + R_2 + R_3$. Therefore, $\Gamma'_{n+2} \cdot l_1 l_2 l_3 - \Gamma_2 \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_{i,1} + P_{i,2} + P_{i,3})$.

By Theorem 4.1, there exists a curve Γ' of degree $\deg(\Gamma') = \deg(\Gamma'_{n+2}) - \deg(\Gamma_2) = n$ such that $\Gamma' \cdot l_1 l_2 l_3 = \sum_{i=1}^{n} (P_{i,1} + P_{i,2} + P_{i,3})$. So, Γ' is an algebraic curve of degree n that passes through the $3n$ points $P_{1,1}, \dots, P_{n,1}, P_{1,2}, \dots, P_{n,2}, P_{1,3}, \dots, P_{n,3}$. This completes the proof. $\square$

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